International Journal of Mathematics, Statistics and Operations Research
Vol. 5; Number 1; 2025, pp. 1-17
https://doi.org/10.47509/IJMSOR.2025.v05i01.01

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Theory and Simulation Study of Entropy Transformed Ailamujia Distribution

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ABSTRACT
This study presents a modified one-parameter Ailamujia distribution called the Entropy Transformed Ailamujia distribution (ETAD). The ETAD properties like order and reliability statistics, entropy, moment and moment generating function, quantile function, and its variability measures are derived. The maximum likelihood technique (MLT) method was used in estimating the parameter of ETAD. Through simulation at different sample sizes of between 20 and 1500 with varying parameter values of 0.1, 0.3, 0.5, and 0.7, the MLT was found to be consistent, efficient, and unbiased for estimating the ETAD parameter.

KEYWORDS
Ailamujia distribution, Entropy Transformation, Technique of Maximum likelihood, Simulation.

1. Introduction

Every model developed has to be sufficient and consistent for handling the problem for which it is created. The parameters of the model have to be estimated using which ever estimation technique that is stable and efficient. The maximum likelihood technique (MLT) has been one of the very popular estimation techniques over the years. It is consistent and efficient especially when the sample size of the data is large. In probability and distribution theory, the MLT is a popular choice among all other contending methods like method of moments, maximum product of spacing, etc. Hence, the consistency and accuracy of statistical analysis in highly influenced by the efficient technique of estimating the parameters of the model or probability distribution adopted [1–5]. Many authors have applied the MLT in estimating the parameters of their developed or modified distributions and they include [6–12], etc. in this study a modified one parameter distribution is introduced using an entropy transformation technique as used by [13]. Also, [14] proposed a probability distribution called the Entropy Transformed Weibull distribution and Entropy Transformed Rayleigh distribution was derived by [15]. This paper examined the MLT stability and efficiency of Entropy Transformed Ailamujia Distribution (ETAD) which is a modification of the

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Article History
Received : 04 January 2025; Revised : 06 February 2025; Accepted : 05 June 2025; Published : 26 June 2025

To cite this paper
Ikwuoche John David, Micheal Inalegwu Ali & Stephen Mathew (2025). Theory and Simulation Study of Entropy Transformed Ailamujia Distribution. *International Journal of Mathematics, Statistics and Operations Research.* 5(1), 1-17.

Ailamujia Distribution introduced by [16].

The remainder of this paper is structured as follows: Section 2 presents some properties of ETAD. The simulation to observe the stability of MLT at different sample sizes is given in Section 3 by applying bias, variance, standard error, and root mean square error estimates for each sample sizes used. Section 4 concludes.

2. Methods

In this section, the ETAD properties derivations and results from simulation at different sample sizes are presented. The Probability Density Function (PDF), Cumulative Distribution Function (CDF), reliability functions, variability measures and distribution shape, measures of uncertainty, and estimation of the ETAD parameter using the MLT through simulation are presented. The estimates of absolute bias, variance, standard error, and root mean square error (RMSE) at different sample sizes and simulated parameter values are obtained to determine the stability of the MLT for estimating the ETAD parameter.

2.1. *The ETAD*

In [17] study, they considered a random variable X which represents the appropriate runtime for any component, and they introduced the following expression

$$g\left(x\right) = F(x) + R\left(x\right) lnR\left(x\right), \; x \geq 0 \tag{1}$$

where,

$$F(x)$$

and

$$R(x)$$

are the cumulative and reliability function, respectively, of a positive continuous random variable X. They called g(x) an "Entropy Transformation" and this possibility is because the term

$$R\left(x\right) lnR\left(x\right)$$

is similar to the associated entropy expression for the density function of a continuous random variable

$$X$$

$$H\left(f\right) = -\int_{0}^{\infty} f\left(x\right) \ln f\left(x\right) dx \tag{2}$$

Differentiating (1) with respect to x the PDF is obtained, its integral gives the CDF, and fixing the range validates the PDF:

$$\int_0^\infty g'(x)dx = \int_0^\infty R'(x)\ln R(x)\,dx = 1 \tag{3}$$

The CDF of the random variable X of Ailamujia distribution (AD) is given by Lv et al. (2002):

$$F(x) = 1 - (1+2\theta x)\,e^{-2\theta x}, x > 0, \theta > 1 \tag{4}$$

The reliability function R(x) of AD is given as

$$R(x) = 1 - F(x) = (1+2\theta x)\,e^{-2\theta x}, x > 0, \theta > 1 \tag{5}$$

Substituting equation (4) and equation (5) into (1) and take the derivative w.r.t x gives

$$g(x) = 1 - (1+2\theta x)\,e^{-2\theta x} + (1+2\theta x)\,e^{-2\theta x}\ln\left((1+2\theta x)\,e^{-2\theta x}\right) \tag{6}$$

However, from equation (3) it can be easily seen that the differentiation of equation (6) can be easily obtained as

$$g'(x) = \left(\frac{d}{dx}\left[(1+2\theta x)\,e^{-2\theta x}\right]\right)\ln\left((1+2\theta x)\,e^{-2\theta x}\right)$$

$$= \left[2\theta e^{-2\theta x} + (1+2\theta x)\,e^{-2\theta x}(-2\theta)\right]\ln\left((1+2\theta x)\,e^{-2\theta x}\right)$$

$$= \left(e^{-2\theta x}[2\theta - 2\theta - 4\theta^2 x]\right)\ln\left((1+2\theta x)\,e^{-2\theta x}\right)$$

$$= -4\theta^2 x e^{-2\theta x}\ln\left((1+2\theta x)\,e^{-2\theta x}\right) \tag{7}$$

Let

$$f(x)$$

be

$$g'(x)$$

then

$$f(x) = -4\theta^2 x e^{-2\theta x}\ln\left((1+2\theta x)\,e^{-2\theta x}\right) \quad x > 0,\, \theta > 1$$

is the PDF of the ETAD. The corresponding CDF after integrating equation (7) is obtained as

$$F(x) = (2\theta x + 1)\, e^{-2\theta x} \left(\ln(2\theta x + 1) - 2\theta x - 1\right) \tag{8}$$

The PDF and CDF plots at different values of θ are presented in Figure 1 and Figure 2, respectively.

2.2. *Linear representation of ETAD*

The ETAD is linearly represented for simplicity using the Taylor series expansion theorem at order four (4). Expanding the ln part of

$$f(x)$$

and let it be represented by

$$t(x),$$

thus

$$\frac{d}{dx}[t(x)] = t'(x) = \frac{d}{dx}\left[\ln\left((2\theta x + 1)\, e^{-2\theta x}\right)\right] = \frac{1}{(2\theta x + 1)\, e^{-2\theta x}} \frac{d}{dx}\left[(2\theta x + 1)\, e^{-2\theta x}\right] \tag{9}$$

$$= \frac{\left(2\theta e^{-2\theta x} - 2\,(2\theta x + 1)\,\theta e^{-2\theta x} \cdot 1\right) e^{2\theta x}}{(2\theta x + 1)\, e^{-2\theta x}} = \frac{\left(2\theta e^{-2\theta x} - 2\theta\,(2\theta x + 1)\, e^{-2\theta x}\right)}{2\theta x + 1} \tag{10}$$

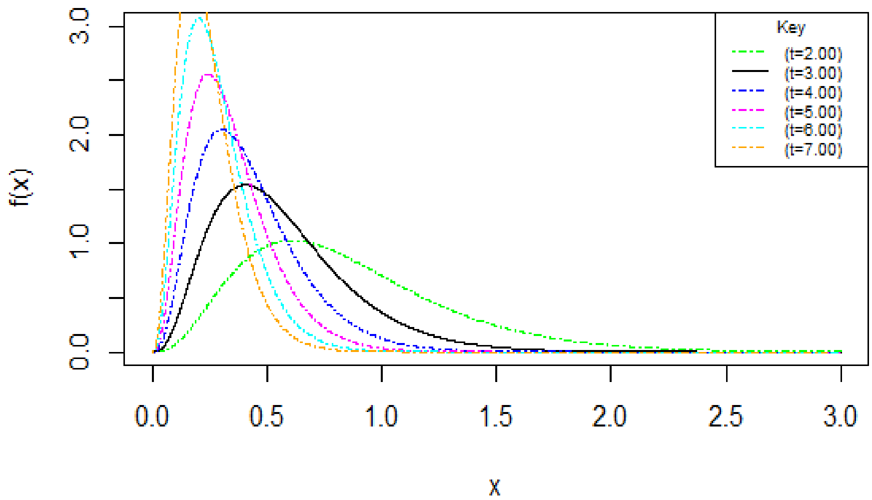

Figure 1. PDF Plots of ETAD

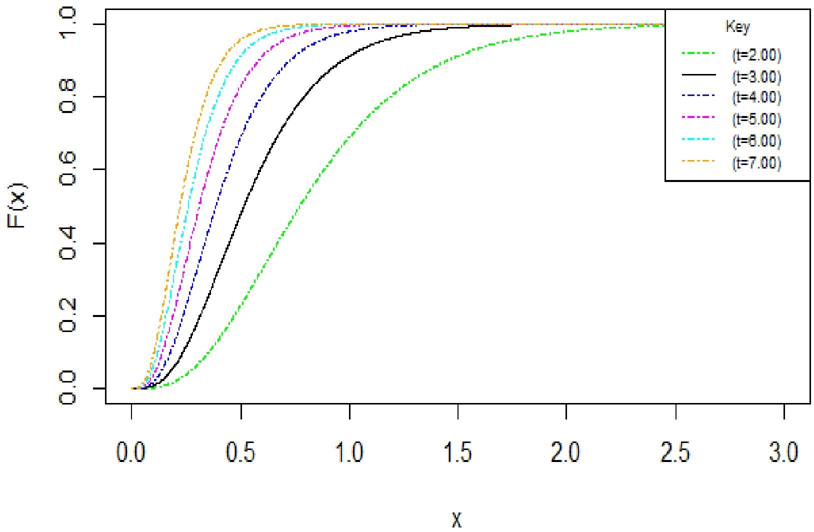

Figure 2. CDF Plots of ETAD

By expansion, equation (10) becomes

$$t'(x) = -\frac{4\theta^2 x}{2\theta x + 1} = 0 \tag{11}$$

$$\frac{d^2}{dx} = t''(x) = \frac{d}{dx}\left[-\frac{4\theta^2 x}{2\theta x + 1}\right] = -4\theta^2 \frac{d}{dx}\left[\frac{x}{2\theta x + 1}\right] = \frac{4\theta^2\left((2\theta x + 1) - 2(2\theta)\right)}{(2\theta x + 1)^2} \tag{12}$$

By expansion, equation (12) becomes

$$t''(0) = \frac{4\theta^2}{(2\theta x + 1)^2} = 4\theta^2 \tag{13}$$

$$\frac{d^3}{dx^3} = t'''(x) = \frac{d}{dx}\left[\frac{-4\theta^2}{(2\theta x + 1)^2}\right] = -4\theta^2 \frac{d}{dx}\left[\frac{1}{(2\theta x + 1)^2}\right] = -4\theta^2(-2)(2\theta x + 1)^{-3}2\theta \tag{14}$$

$$= \frac{16\theta^3}{(2\theta x + 1)^3} \tag{15}$$

By expansion, equation (15) becomes

$$t'''(0) = 16\theta^3 \tag{16}$$

$$\frac{d^4}{dx^4} = t''''(x) = \frac{d}{dx}\left[\frac{16\theta^3}{(2\theta x+1)^3}\right] = 16\theta^3\frac{d}{dx}\left[\frac{1}{(2\theta x+1)^3}\right] \tag{17}$$

$$= 16\theta^3(-3)(2\theta x+1)^{-4}2\theta = -\frac{96\theta^4}{(2\theta x+1)^4} \tag{18}$$

By expansion, equation (18) becomes

$$t''''(\theta) = 96\theta^4 \tag{19}$$

Therefore,

$$t(0) = \frac{0x^0}{0!} + \frac{0x^1}{1!!} - 4\theta^2\frac{x^2}{2!} + 16\theta^3\frac{x3}{3!} - 96\theta^4\frac{x^4}{4!} + \cdots \sum_{n=2}^{\infty} t(0) = -2\theta^2x^2 + 2\cdot 67\theta^3x^3 - 4\theta^4x^4 + \cdots \tag{20}$$

The approximate and nth series expansion of equation (20) is given, respectively as

$$\sum_{n=2}^{\infty} t(0) = -2\theta^2x^2 + 3\theta^3x^3 - 4\theta^4x^4 + \cdots \tag{21}$$

and

$$\sum_{n=2}^{\infty} t(0) = \sum_{n=2}^{\infty} (2)(1.5)^{(n-2)}(-1)^{(n+1)}\theta^n x^n. \tag{22}$$

Therefore, the linear representation of ETAD is given as

$$f(x) = -4\theta^2 x e^{-2\theta x}\left[\sum_{n=2}^{\infty}(2)(1.5)^{(n-2)}(-1)^{(n+1)}\theta^n x^n\right]$$

$$= -4\theta^2 x e^{-2\theta x}\left[G_n\theta^n x^n\right] = -4\theta^{2+n}G_n x^{1+n}e^{-2\theta x} \tag{23}$$

where, $G_n = \sum\limits_{n=2}^{\infty}(2)(1.5)^{(n-2)}(-1)^{(n+1)}$.

2.3. *Some Properties of ETAD*

The survival [S(x)], hazard [h(x)], and odd [O(x)] functions are presented as well as their respective plots follows.

$$S(x) = 1 - F(X \le x) = 1 - (2\theta x+1)e^{-2\theta x}(\ln(2\theta x+1) - 2\theta x - 1) \tag{24}$$

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} = \frac{-4\theta^2 x\left[\ln(2\theta x+1)e^{-2\theta x}\right]}{1-(2\theta x+1)e^{-2\theta x}(\ln(2\theta x+1)-2\theta x-1)} \tag{25}$$

$$O(x)=\frac{F\left(x\right)}{1-F\left(x\right)}=\frac{F(x)}{S(x)}=\frac{\left(2\theta x+1\right)e^{-2\theta x}\left(\ln\left(2\theta x+1\right)-2\theta x-1\right)}{1-\left(2\theta x+1\right)e^{-2\theta x}\left(\ln\left(2\theta x+1\right)-2\theta x-1\right)} \tag{26}$$

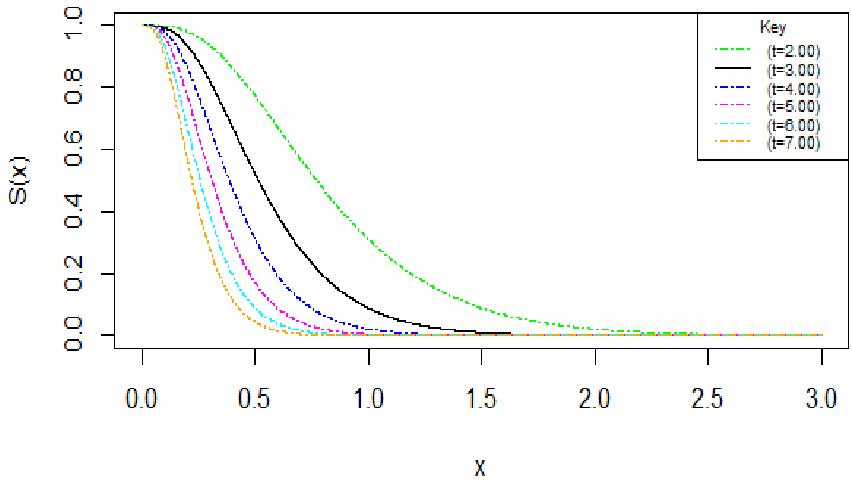

Figure 3. Plot of S(x) at Different Parameter Values

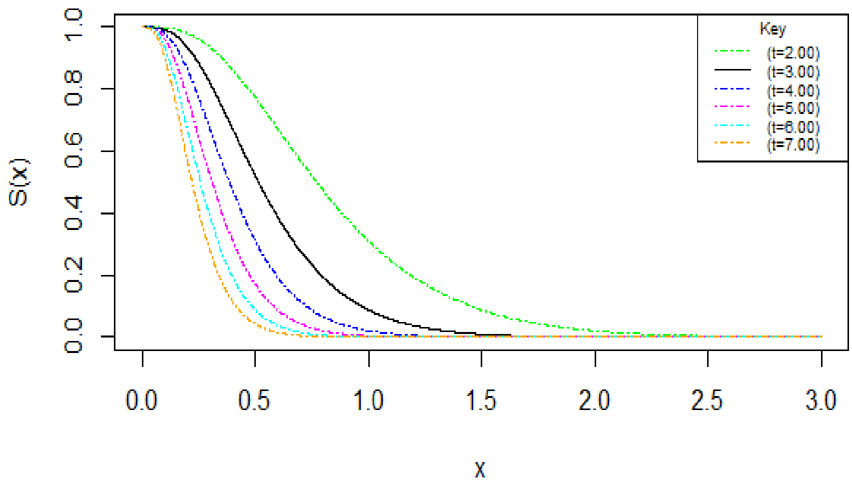

Figure 4. Plot of H(x) at Different Parameter Values

Quantile Function [QF]: the inverse of the CDF of ETAD gives the QF of ETAD, through the Lambert W function transformation and it is expressed as;

$$x = \frac{-W_{-1}\left[-10W_0\left(\frac{u}{10}\right)e^{-1}\right]-1}{2\theta} \tag{27}$$

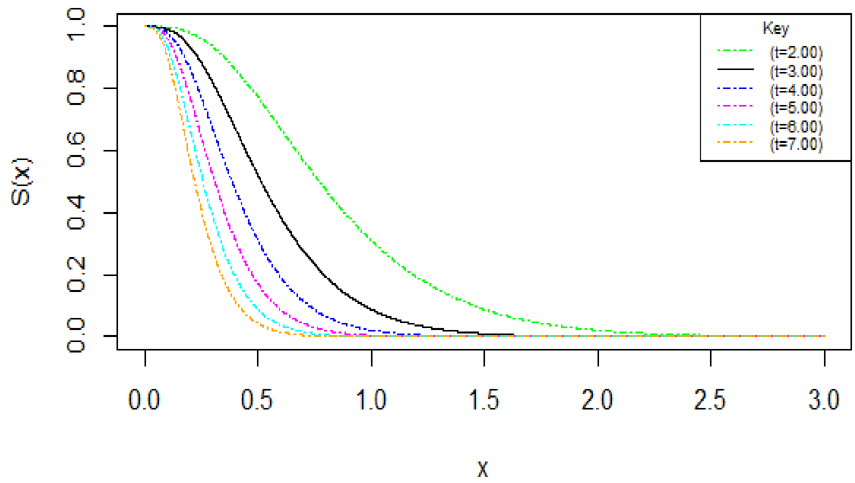

Figure 5. Plot of O(x) at Different Parameter Values

2.4. *Moments and Moment Generating Function*

Moments: let the random variable $X \sim ETAD(x, \theta)$, the r^{th} moment(μ_r), can be obtained as

$$\mu^r = E(x^r) = \int_0^\infty x^r f(x, \theta)\, dx \tag{28}$$

$f(x) = -4\theta^{2+n} G_n x^{1+n} e^{-2\theta x}$.

$\mu_r = \int_0^\infty x^r \cdot -4\theta^{2+n} x^{n+1} G_n e^{-2\theta x} dx = -4\theta^{2+n} G_n \int_0^\infty x^{n+1+r} e^{-2\theta x} dx.$

Let $u = 2\theta x$; $\frac{du}{dx} = 2\theta \Rightarrow du = \frac{1}{2\theta} du$; $dx = \frac{du}{2\theta}$.

Where $x = 0$, $u = 2\theta \times 0 = 0$; As $x \to \infty$, $u \to \infty$. By substitution the following is obtained:

$$\mu_r = -4\theta^{2+n} G_n \int_0^\infty \left(\frac{u}{2\theta}\right)^{n+1+r} e^{-u} \frac{du}{2\theta}$$

$$= -4G_n\theta^{2+n} (2\theta)^{-(n+1+r)} \int_0^\infty u^{n+1+r} e^{-u} du$$

$$= -4G_n\theta^{2+n} (2\theta)^{-(n+1+r)} \Gamma(n+1+r) \tag{29}$$

$$\mu_r = -4G_n\theta^{2+n}[2\theta]^{-(n+r+2)}\Gamma(n+r+2). \tag{30}$$

The mean of ETAD, that is, E[x] is given as: $\mu_r = -4G_n\theta^{2+n}[2\theta]^{-(n+r+2)}\Gamma(n+r+2)$

$$E(x) = \mu_1 = -4G_n\theta^{2+n}[2\theta]^{-[n+1+2]}\Gamma(n+1+2) = -\left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\right] \tag{31}$$

The population variance, σ^2 of ETAD can be obtained as follows:

$$Var(x) = E\left[x^2\right] - [E(x)]^2 \tag{32}$$

$$E[x]^2 = \mu_2 = -4G_n\theta^{2+n}[2\theta]^{-(n+4)}$$

$$= -4G_n\theta^{2+n}2^{-(n+4)}\theta^{-(n+4)}\Gamma(n+4)$$

$$= -2^2 \cdot 2^{-(n+4)}G_n\theta^{2+n}\theta^{-(n+4)}\Gamma(n+4)$$

$$= -\left[2^{-(n+2)}G_n\theta^{-2}\Gamma(n+4)\right] = -\left[\frac{G_n\Gamma(n+4)}{2^{(n+2)}\theta^2}\right] = -\frac{G_n\Gamma(n+4)}{2^{(n+2)}\theta^2} - \left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\right]^2$$

$$Var(x) = \frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta} - \frac{G_n\Gamma(n+4)}{2^{(n+2)}\theta^2} = \frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1 - \frac{(n+3)}{2\theta}\right] \tag{33}$$

The Coefficient of Variance (CV) of ETAD can be obtained as follows:

$$CV = \frac{\sigma}{\mu}100\% = \frac{\sqrt{\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1 - \frac{(n+3)}{2\theta}\right]}}{\frac{-G_n\Gamma(n+3)}{2^{(n+1)}\theta}}$$

$$= \left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1 - \frac{(n+3)}{2\theta}\right]\right]^{\frac{1}{2}} \frac{\left(-2^{(n+1)}\theta\right)}{G_n\Gamma(n+3)} \tag{34}$$

$$= \left[\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\right]^{\frac{1}{2}} \left[\frac{-2^{(n+1)}\theta}{G_n\Gamma(n+3)}\right] \left[1 - \frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}$$

$$= -[G_n\Gamma(n+3)]^{\frac{1}{2}} \left[2^{(n+1)}\theta\right] [G_n\Gamma(n+3)]^{-1} \left[1 - \frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}$$

$$= -[G_n\Gamma(n+3)]^{\frac{1}{2}} \left[2^{(n+1)}\theta\right]^{\frac{1}{2}} \left[1 - \frac{(n+3)}{2\theta}\right]^{\frac{1}{2}} = -\left[\frac{2^{(n+1)}\theta}{G_n\Gamma(n+3)}\right]^{\frac{1}{2}} \left[1 - \frac{(n+3)}{2\theta}\right]^{\frac{1}{2}}$$

$$CV = -\left[\frac{2^{(n+1)}\theta}{G_n\Gamma(n+3)} \left[1 - \frac{(n+3)}{2\theta}\right]\right]^{\frac{1}{2}} \times 100\% \tag{35}$$

The Harmonic Mean (HM) of ETAD is obtained as follows:

$$HM(X) = \int_0^\infty \frac{1}{x} f(x)\,dx \tag{36}$$

$$= \int_0^\infty \frac{1}{x}\left(-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}\right)dx$$

$$= -4G_n\theta^{2+n}\int_0^\infty x^{-n}x^{1+n}e^{-2\theta x}dx$$

$$= -4G_n\theta^{2+n}\int_0^\infty x^{n}e^{-2\theta x}dx$$

$$= -4G_n\theta^{2+n}\left[\frac{n!}{(2\theta)^{n+1}}\right] = -\frac{-4G_n\theta^{2+n}n!}{2^{n+1}\theta^{n+1}}$$

$$= -2^2 \cdot 2^{-(n+1)}G_n\theta^{2+n}\theta^{-(n+1)}n! = 2^{(1-n)}G_n\theta n! \tag{37}$$

The Mode (M) of ETAD can be obtained as follows:

$$M(x) = \frac{d}{dx}\ln\left(f(x,\theta)\right) \tag{38}$$

$$= \frac{d}{dx}\ln\left[-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}\right] = \frac{1}{-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}}\frac{d}{dx}\left[-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}\right]$$

$$= \frac{\left[-4G_n\theta^{2+n}\frac{d}{dx}\left[x^{1+n}e^{-2\theta x}\right]\right]}{-4G_n\theta^{2+n}x^{1+n}e^{-2\theta x}} = \frac{(n+1)x^ne^{-2\theta x} + x^{n+1}\cdot e^{-2\theta x}(-2\theta)}{x^{n+1}e^{-2\theta x}}$$

$$= \frac{(n+1)x^ne^{-2\theta x} - 2x^{n+1}\cdot e^{-2\theta x}\theta}{x^{n+1}e^{-2\theta x}} = \frac{x^ne^{-2\theta x}\left[(n+1) - 2x\theta\right]}{x^n\cdot x\cdot e^{-2\theta x}} = \frac{(n+1)-2\theta x}{x}$$

$$= \left[(n+1) - 2\theta x\right]x^{-1} = M(x) = (n+1)x^{-1} - 2\theta \tag{39}$$

The Median can be derived by using equation (27) which is the derived quantile function of ETAD and fixing$u = 0.5$, the median of ETAD is obtained as follows:

$$x = \frac{-W_{-1}\left[-10W_0\left(\frac{0.5}{10}\right)e^{-1}\right] - 1}{2\theta} = \frac{-W_{-1}\left[-10W_0(0.05)e^{-1}\right] - 1}{2\theta} \tag{40}$$

Skewness [SK]: The skewness of a probability distribution is a measure of symmetry and the lack of symmetry of the probability distribution. The ETAD SK is derived from the 3rd moment of the mean.

$$SK = \frac{E\left(x^3\right) - 3\mu\sigma^2 - \mu^3}{\sigma^3} \tag{41}$$

$$SK = \frac{2^{\left[\frac{3n+7}{2}\right]}\left[\frac{G_n\Gamma n+5}{2^{(n+2)}} + \frac{3[G_n\Gamma n+3]^2[2(n+3)+G_n\Gamma n+3]}{2^{2n+3}} - \frac{[G_n\Gamma n+3]^3}{2^{3n+2}}\right]}{\sqrt{i[G_n\Gamma n+3\left[2\left(n+3\right)+G_n\Gamma n+3\right]]}^{3/2}} \tag{42}$$

Kurtosis [KT]: The kurtosis measure whether or not the probability distribution is heavy-tailed. For ETAD the KT is derived from the 4th moment and is obtained as

$$KT = \frac{E\left(x^4\right) - 4\mu E\left[x^3\right] + 6\sigma^2 E\left[x^2\right] - 3E\left[x^4\right]}{\sigma^4} \tag{43}$$

$$KT = \frac{3G_n\Gamma\left(n+4\right)\left[2\left(n+3\right)+G_n\Gamma\left(n+3\right)\right] - 4G_n\Gamma\left(n+5\right) + 2^{(n+2)}\left(n+5\right)\left(n+4\right)\left(n+3\right)}{G_n\Gamma\left(n+3\right)\left[2\left(n+3\right)+G_n\Gamma\left(n+3\right)\right]^2} \tag{44}$$

Dispersion Index [DI]: It is simply defined as the variance divided by the mean. It tells how dispersed the mean is from the variance. The DI for the ETAD is obtained as

$$DI = \frac{\sigma^2}{\mu} = \frac{\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}\left[1 - \frac{(n+3)}{2\theta}\right]}{\frac{G_n\Gamma(n+3)}{2^{(n+1)}\theta}} = 1 - \frac{(n+3)}{2\theta} \tag{45}$$

Moment Generating Function [MGF]: The MGF of X, say$M_X\left(t\right)$, is

$$M_x\left(t\right) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f\left(x,\theta\right)dx = \int_0^\infty e^{tx}\cdot -4\theta^{2+n}G_n x^{1+n}e^{-2\theta x}dx; e^{tx} = \sum_{j=0}^\infty \frac{(tx)^m}{m!} \tag{46}$$

By substitution equation (46) becomes

$$M_x\left(t\right) = \int_0^\infty \sum_{j=0}^\infty \frac{(tx)^m}{m!}\cdot -4\theta^{2m}G_n x^{1+n}e^{-2\theta x}dx \tag{47}$$

$$= \int_0^\infty \sum_{j=0}^\infty \frac{t^m}{m!}x^m \cdot -4\theta^{2+n}G_n x^{1+n}e^{-2\theta x}dx = \phi\int_0^\infty x^{(1+n+m)}\theta^{2+n}e^{-2\theta x}dx \tag{48}$$

Let $u = 2\theta x \Rightarrow x = \frac{u}{2\theta}$; $\frac{d}{dx} = 2\theta \Rightarrow dx = \frac{1}{2\theta}dx$. Where, $x = 0, u = 0, x \to \infty, u \to \infty$.

Substituting into equation (48) to get

$$= \varphi \int_0^\infty \left(\frac{u}{2\theta}\right)^{(1+n+m)} \theta^{2+n} e^{-u} \frac{du}{2\theta} = \varphi \int_0^\infty u^{(1+n+m)} (2\theta)^{-[1+n+m]} \theta^{2+n} e^{-u} 2\theta^{-1} du$$

$$= \phi(2\theta)^{-[2+n+m]} \theta^{2+n} \int_0^\infty u^{[1+n+m]-1} e^{-u} du$$

$$M_x(t) = \phi[2\theta]^{-[2+n+m]} \theta^{2+n} \Gamma[2+n+m] \tag{49}$$

2.5. *Entropy*

Suppose a random variable X follows the ETA distribution, then the Renyi entropy (Renyi, 1961) which measures the uncertainty of information is expressed as

$$I_R(c) = \frac{1}{1-c} \log \int_0^\infty f^c(x) dx$$

$$= \frac{1}{1-c} \log \left[\int_0^\infty \left\{ -4\theta^2 x e^{-2\theta x} \left[\log\left(e^{-2\theta x} (2\theta x + 1) \right) \right] \right\} \right]^c dx \tag{50}$$

$$= \frac{1}{1-c} \log \left[\int_0^\infty \left(-4\theta^2\right)^c x^c e^{-2\theta c x} \left[\log\left(e^{-2\theta c x} [2\theta x + 1] \right) \right]^c \right] dx$$

$$= \frac{1}{1-c} \log \left[\int_0^\infty \left[-4\theta^{2+n} G_n x^{1+n} e^{-2\theta x} \right]^c dx \right]$$

$$= \frac{1}{1-c} \log \left[\int_0^\infty \left[-4\theta^2 G_n \right]^c x^{c(1+n)} e^{-2\theta c x} dx \right] = \frac{1}{1-c} \log \left[\left[-4\theta^{2n} G_n \right]^c \int_0^\infty x^{c(1+n)} e^{-2\theta x} dx \right]$$

Let $u = 2\theta c x, x = \frac{u}{2\theta c}$. $\frac{d}{dx} = 2\theta c, du = \frac{du}{2\theta c}$. Where $x = 0, u = 0, x \to \infty, u \to \infty$.

Therefore,

$$I_R(c) = \frac{1}{1-c} \log \left[\left(-4\theta^{2+n} G_n \right)^c \int_0^\infty \left(\frac{u}{2\theta c} \right)^{c(1+n)} e^{-u} \frac{du}{2\theta c} \right]$$
$$= \frac{1}{1-c} \log \left[\left(-4\theta^{2+n} G_n \right)^c \int_0^\infty u^{c(1+n)} (2\theta c)^{-(n+1)} (2\theta c)^{-1} e^{-u} du \right]$$
$$= \frac{1}{1-c} \log \left[\left(-4\theta^{2+n} G_n \right)^c (2\theta c)^{-(n+1)^{c-1}} \int_0^\infty u^{c(1+n)} e^{-u} du \right]$$
$$= \frac{1}{1-c} \log \left[\left(-4\theta^{2+n} G_n \right)^c (2\theta c)^{-(n+1)c-1} \int_0^\infty u^{c(1+n)+1-1} e^{-u} du \right]$$

$$= \frac{1}{1-c} \log \left[\left(-4\theta^{2+n} G_n \right)^c (2\theta c)^{-c(n+1)-1} \Gamma\left(c[1+n] + 1 \right) \right]$$

$$= \frac{1}{1-c} \log \left[\left(-4\theta^{2+n} G_n \right)^c \right] + \log \left[(2\theta c)^{-c(n+1)-1} \right] + \log \left[\Gamma \left(c \left(n+1 \right) + 1 \right) \right] \tag{51}$$

2.6. *Order statistics*

Given an independent characteristic distribution of a random sample$X_1; X_2 \cdots ; X_n$, this sample can be in ordered form as $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. These representations are called order statistics. The 1^{st} order statistics,$X_{(1)}$ is the minimum, the 2^{nd} order statistics, $X_{(2)}$ is the second while $X_{(n)}$ is the n^{th} order statistics and the maximum. Suppose a random sample $X_1, X_2 \cdots , X_n$ from the ETAD is ordered as$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$, then the PDF$f_{(n;n)}(X)$ of the k^{th} order statistics can be define as:

$$f_{(k,n)}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x) \times F(x)^{k-1} \times [1 - F(x)]^{n-k} \tag{52}$$

where, $F(X)$ and $f(X)$ are the CDF and PDF respectively, of the ETAD. Expanding $[1 - F(x)]^{n-k}$ binomially and substitute the expression, CDF, and PDF into equation (52) gives;

$$f_{(k,n)}(x) = \sum_{c_5}^{\infty} \tfrac{n!(-1)^{c_5}}{(k-1)!(n-k-c_5)!c_5!} \left(-4\theta^2 x e^{-2\theta x} \ln \left[(2\theta x + 1) e^{-2\theta x} \right] \right) \begin{pmatrix} n-k \\ c_5 \end{pmatrix} (-1)^{c_5}$$
$$\times \left[e^{-2\theta x} (2\theta x + 1) \left\{ \ln \left[2\theta x + 1 \right]^{-2\theta x} \right\} - 1 \right]^{c_5 + k - 1} \tag{53}$$

Putting k = 1 and k = n in equation (53), the 1^{st}or minimum order statistic and n^{th} or maximum order statistic are obtained as;

$$\begin{aligned} f_{(1,n)}(x) = \sum_{c_5}^{\infty} \tfrac{n!(-1)^{c_5}}{(n-1-c_5)!c_5!} \left\{ -4\theta^2 x e^{-2\theta x} \ln \left[(2\theta x + 1)^{-2\theta x} \right] \right\} \\ \times \left[e^{-2\theta x} (2\theta x + 1) \left\{ \ln \left[(2\theta x + 1)^{-2\theta x} \right] - 1 \right\} \right]^{c_5} \end{aligned} \tag{54}$$

$$\begin{aligned} f_{(n,n)}(x) = \sum_{c_5}^{\infty} \tfrac{n!(-1)^{c_5}}{(n-1)!-c_5!c_5!} \left\{ -4\theta^2 x e^{-2\theta x} \ln \left[(2\theta x + 1) e^{-2\theta x} \right] \right\} \\ \times \left[e^{-2\theta x} (2\theta x + 1) \left\{ \ln \left[(2\theta x + 1) e^{-2\theta x} \right] - 1 \right\} \right]^{c_5 + n - 1} \end{aligned} \tag{55}$$

2.7. *MLT for ETAD Parameter Estimation*

Suppose X_1, X_2, ..., X_n is a random sample with size n drawn from ETAD. The likelihood function is given as follows:

$$L(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^{n} f(x; \theta) \tag{56}$$

$$= \prod_{i=1}^{n} \left[-4\theta^2 x e^{-2\theta x} \left[\ln \left[(2\theta x + 1) e^{-2\theta x} \right] \right] \right]$$

$$= \left[-4\theta^2\right]^n \prod_{i=1}^{n} x \prod_{i=1}^{n} e^{-2\theta x} \prod_{i=1}^{n} \left[\ln\left[(2\theta x+1)\, e^{-2\theta x}\right]\right] \tag{57}$$

The log-likelihood (log L) of the ETAD becomes

$$\log L\left(x_1 : x_n; \theta\right) = \log\left[\left(-4\theta^2\right)^n \prod_{i=1}^{n} x \prod_{i=1}^{n} e^{-2\theta x} \prod_{i=1}^{n} \left[\ln\left[(2\theta x+1)\, e^{-2\theta x}\right]\right]\right] = Z \tag{58}$$

$$Z = n\left(-4\theta^2\right) + \sum_{i=1}^{n} \log\left(x_i\right) - 2\theta \sum_{i=1}^{n} x_i + \log\left[\prod_{i=1}^{n} \left[\ln\left[(2\theta x+1)\, e^{-2\theta x}\right]\right]\right]$$

$$Z = -4n\theta^2 + \sum_{i=1}^{n} \log\left(x_i\right) - 2\theta \sum_{i=1}^{n} x_i + \left[\sum_{i=1}^{n} \log\left[\ln\left[(2\theta x+1)\, e^{-2\theta x}\right]\right]\right] \tag{59}$$

The parameter estimate $\hat{\theta}$ of ETAD is obtained by differentiating equation (59) partially with respect to θ and equate to zero. This implies,

$$\frac{\partial Z}{\partial \theta} = -8n\theta - 2\sum_{i=1}^{n} x_i - 4\sum_{i=1}^{n} \left[\frac{\theta^2 x}{(2\theta x+1)\ln\left[\left((2\theta x+1)\, e^{-2\theta x}\right)\right]}\right] = 0 \tag{60}$$

$$\sum_{i=1}^{n} x_i = -4n\theta - 2\sum_{i=1}^{n} \left[\frac{\theta^2 x}{(2\theta x+1)\ln\left[\left((2\theta x+1)e^{-2\theta x}\right)\right]}\right]$$

$$\hat{\theta} = -\left(4\bar{x}_i + \frac{1}{2n}\sum_{i=1}^{n} \left[\frac{\hat{\theta}^2 x_i}{\left(2\hat{\theta} x_i+1\right)\ln\left[\left(\left(2\hat{\theta} x_i+1\right) e^{-2\hat{\theta} x_i}\right)\right]}\right]\right) \tag{61}$$

Equation (61) clearly reveal that there is no close form expression for θ and this implies it can only be estimated through numerical methods using software like SAS, R, Maple, etc., for this research the R open source software was used.

3. Applications

In this section, a simulation study for determining the stability of MLT in estimating the parameter of ETAD is presented. The simulation study are carried out using `R` software [18] for fitting the ETAD at a small sample size of 20 to a large sample size of 1500.

3.1. *Simulation study*

The usefulness of simulation is to determine the efficiency and consistency of the estimate of MLT. Using different parameters values and sample sizes (20-1500), the estimation methods used for the comparing are based on absolute bias (AB), variance (S^2), standard error (SE), and root mean square error (RMSE). Their formulas are given in (62), (63), (64), and (65), respectively.

Table 1. The AB, S^2, SE, and RMSE Values at Different SS and PV

SS	**PV**	**Estimate**	**AB**	S^2	**SE**	**RMSE**
20	0.1	0.1613	0.0613	0	0	0.0038
20	0.3	0.5012	0.2012	0.0003	0.0039	0.0408
20	0.5	0.8389	0.3389	0.0008	0.0063	0.1157
20	0.7	1.1764	0.4764	0.0015	0.0087	0.2285
40	0.1	0.1609	0.0609	0	0	0.0037
40	0.3	0.4911	0.1911	0.0001	0.0016	0.0366
40	0.5	0.8204	0.3204	0.0004	0.0032	0.1031
40	0.7	1.1497	0.4497	0.0008	0.0045	0.203
100	0.1	0.1608	0.0608	0	0	0.0037
100	0.3	0.4856	0.1856	0.0001	0.001	0.0345
100	0.5	0.8102	0.3102	0.0002	0.0014	0.0964
100	0.7	1.1347	0.4347	0.0003	0.0017	0.1893
250	0.1	0.1606	0.0606	0	0	0.0037
250	0.3	0.4831	0.1831	0	0	0.0335
250	0.5	0.8055	0.3055	0	0	0.0933
250	0.7	1.1279	0.4279	0.0001	0.0006	0.1832
500	0.1	0.1606	0.0606	0	0	0.0037
500	0.3	0.4824	0.1824	0	0	0.0333
500	0.5	0.8042	0.3042	0	0	0.0925
500	0.7	1.126	0.426	0	0	0.1815
750	0.1	0.1605	0.0605	0	0	0.0037
750	0.3	0.482	0.182	0	0	0.0331
750	0.5	0.8035	0.3035	0	0	0.0921
750	0.7	1.125	0.425	0	0	0.1806
1000	0.1	0.1605	0.0605	0	0	0.0037
1000	0.3	0.4819	0.1819	0	0	0.0331
1000	0.5	0.8033	0.3033	0	0	0.092
1000	0.7	1.1246	0.4246	0	0	0.1803
1500	0.1	0.1605	0.0605	0	0	0.0037
1500	0.3	0.4818	0.1818	0	0	0.0331
1500	0.5	0.803	0.303	0	0	0.0918
1500	0.7	1.1243	0.4243	0	0	0.18

$$AB = \frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_i\right) \tag{62}$$

$$RMSE = \sqrt{\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i - \theta_i\right)^2} \tag{63}$$

$$S^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} \tag{64}$$

$$SE = \frac{\sigma}{\sqrt{n}} \tag{65}$$

Table 1 presents the simulation results from 20 Sample Sizes (SS) to 1500 SS at different parameter values (PV). The results demonstrate that the MLT is consistent and efficient. This is due to the fact that as the sample size increases from 20 to 1500, the variance, S^2 and standard error, SE decreases and tends to zero. Also, as the SS increases from 20 to 1500, the AB and RMSE values reduces as the parameter estimates increases. However, at constant subsample size as the parameter θ is varied, the θ estimates, AB, S^2, SE and RMSE increases. It is also observed that as the sample size keep increasing the estimates $\hat{\theta}$ tends to θ.

4. Concluding remarks

This study has been able to utilize a novel approach known as the entropy transformation (ET) in modifying a distribution without adding any extra parameter to the original distribution. Illustratively, the Ailamujia distribution was modified using the ET to form ETAD and it was found to be a consistent and stable distribution through the simulation of estimating its parameter with the method of MLT at different sample sizes and parameter values. The different density plots showed that the distribution can comfortably handle normal and right skewed data well. Its application to real life data will be implemented in future research by the authors to compare its capability with other one-parameter distributions using different data sets.

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